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Elementary Potentials and Galerkin's Matrix for an Ellipsoidal Domain in the Recovery of the Gravity Field

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1. Weak Solution - in Terms of Bases

When following this concept, we start with the ***bilinear form***

$$A(u,v) = \sum_{i=1}^{3} \int_{\Omega_{ell}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\mathbf{x} \;, \quad \text{where}$$

$x_i,\ i=1,2,3$ are rectangular ***Cartesian co-ordinates***;

Ω_{ell} … is the ***exterior of an oblate ellipsoid of revolution*** of semiaxes a and b, $a \geq b$, whose center is in the origin, and its axis of rotation coincides with the x_3- axis.

u, v … are ***functions from Sobolev's space*** $W_2^{(1)}(\Omega_{ell})$.

In case that f is a ***function from Lebesgue's space*** $L_2(\partial\Omega_{ell})$, we know that the ***integral identity***

$$A(u,v) = \int_{\partial\Omega_{ell}} v f \; dS \quad \text{valid for all} \quad v \in W_2^{(1)}(\Omega_{ell})$$

defines the function u ***uniquely*** and that u is the ***weak solution of Neumann's BVP***.

The integral identity above is a starting point for ***numerical solution of Neumann's BVP***. Indeed, one can approximate u by

$$u_n = \sum_{j=1}^{n} c_j^{(n)} v_j \ ,$$

v_j are ***members of a function basis*** of $W_2^{(1)}(\Omega_{ell})$ and can look for the ***coefficients*** $c_j^{(n)}$ by solving the respective ***Galerkin system***

$$\sum_{j=1}^{n} c_j^{(n)} A(v_j, v_k) = \int_{\partial\Omega_{ell}} v_k \ f \ dS \ , \qquad k = 1, \ldots, n$$

The justification follows from the strong convergence of u_n to u as $n \to \infty$, i.e. ***in the norm*** $\|\,.\,\|_1$ ***of Sobolev's space*** $W_2^{(1)}(\Omega_{ell})$.

In the sequel we take for v_j, $j = 1, \ldots, n$, radial symmetric functions (***elementary potentials***)

$$v_j(\boldsymbol{x}) = \frac{1}{|\,\boldsymbol{x} - \boldsymbol{y}_j\,|} \quad \text{with} \quad \boldsymbol{y}_j \notin \Omega_{ell}$$

In connection with Ω_{ell} it is natural to use ***ellipsoidal coordinates*** $u,\ \beta,\ \lambda$. They are related to $x_1,\ x_2,\ x_3$ by the equations

$$x_1 = \sqrt{u^2 + E^2}\cos\beta\cos\lambda\,, \quad x_2 = \sqrt{u^2 + E^2}\cos\beta\sin\lambda\,, \quad x_3 = u\sin\beta$$

where $E = \sqrt{a^2 - b^2}$. ***Note*** that $\partial\Omega_{ell}$ is defined by $u = b$.

Following ***Hobson*** (***1931***) and explanations in ***Hotine*** (***1969***), we can express $v_j(\boldsymbol{x})$ in terms of $u,\ \beta,\ \lambda$:

$$v_j(\boldsymbol{x}) = \frac{i}{E}\sum_{n=0}^{\infty}(2n+1)\Big[Q_n(z)P_n(z_j)P_n(\sin\beta)P_n(\sin\beta_j) +$$

$$+\,2\sum_{m=1}^{n}(-1)^m\left(\frac{(n-m)!}{(n+m)!}\right)^2 Q_{nm}(z)P_{nm}(z_j)P_{nm}(\sin\beta)P_{nm}(\sin\beta_j)\cos m(\lambda-\lambda_j)\Big]$$

where P_{nm} and Q_{nm} are ***Legendre functions of the 1st and the 2nd kind***, respectively; while $z = iu/E$, $z_j = iu_j/E$ and $i = \sqrt{-1}$. We suppose $u_j < u$, which guarantees the convergence of the series.

2. Elements of Galerkin's Matrix

Recall first that Greens' identity alternatively yields

$$\longrightarrow \qquad A_{ell}(v_j, v_k) = -\int_{\partial\Omega_{ell}} v_j \frac{\partial v_k}{\partial n_e}\, dS$$

where $dS = a\sqrt{b^2 + E^2 \sin^2\beta}\cos\beta\, d\beta\, d\lambda$ **is the surface element of our ellipsoid. It is clear that for** $u = b$ ***we need the derivative of*** Q_{nm} ***in the direction of its outer unit normal n***$_e$**. Referring to** ***Holota (2001)*****, we can write**

$$\frac{\partial Q_{nm}(z_0)}{\partial n_e} = -\frac{b(n+1)Q^*_{nm}(z_0)}{a\sqrt{b^2 + E^2\sin^2\beta}}\,,$$

where

$$Q^*_{nm}(z_0) = Q_{nm}(z_0) + \frac{n-m+1}{z_0(n+1)} Q_{n+1,m}(z_0)$$

and $z_0 = ib/E$.

The apparatus we prepared makes it possible to compute that

$$A_{ell}(v_j, v_k) = -\frac{4\pi b}{E^2}\sum_{n=0}^{\infty}(n+1)(2n+1)\Big[A_{n0jk}P_n(\sin\beta_j)P_n(\sin\beta_k) + \\ +2\sum_{m=1}^{n}\left(\frac{(n-m)!}{(n+m)!}\right)^3 A_{nmjk}P_{nm}(\sin\beta_j)P_{nm}(\sin\beta_k)\cos m(\lambda_j-\lambda_k)\Big] \tag{1}$$

where

$$A_{nmjk} = Q_{nm}(z_0)Q^{*}_{nm}(z_0)P_{nm}(z_j)P_{nm}(z_k)$$

and $u_j < b$, $u_k < b$ **according to our assumption.**

3. Approximation of the Elements

The problem now is to express the factors A_{nmjk}. We need a *representation of* the functions P_{nm} and Q_{nm}. It turned out that the following formulas

$$P_{nm}(z) = \frac{(2n)!}{2^n n!(n-m)!}(z^2-1)^{\frac{n}{2}} F\left(-\frac{n-m}{2}, -\frac{n+m}{2}, -\frac{2n-1}{2}; \frac{1}{1-z^2}\right)$$

$$Q_{nm}(z) = (-1)^m \frac{2^n n!(n+m)!}{(2n+1)!}(z^2-1)^{-\frac{n+1}{2}} F\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, \frac{2n+3}{2}; \frac{1}{1-z^2}\right)$$

where $F(\alpha, \beta, \gamma; x)$ is the *hypergeometric function*, are the most suitable, see *Holota* (*2001*).

Passing to *hypergeometric series*, we can compute that

$$A_{nmjk} = \frac{1}{(2n+1)^2}\left[\frac{(n+m)!}{(n-m)!}\right]^2 \frac{1}{z_0^2-1} q^n \left[1+\frac{(n+m+1)(n-m+1)}{2n+3}\frac{1}{1-z_0^2}+\right.$$

$$\left.+\frac{(n+m+1)(n-m+1)}{(n+1)(2n+3)}\frac{1}{z_0\sqrt{z_0^2-1}}-\frac{(n-m)(n+m)}{2(2n-1)}\left(\frac{1}{1-z_j^2}+\frac{1}{1-z_k^2}\right)+\cdots\right]$$

where

$$q=\frac{\sqrt{z_j^2-1}\sqrt{z_k^2-1}}{z_0^2-1}=\frac{\sqrt{u_j^2+E^2}\sqrt{u_k^2+E^2}}{a^2}$$

For u_j, u_k close to b one can deduce that within an accuracy in brackets up to terms multiplied by e^2

$$A_{nmjk} = -\frac{E^2}{a^2}\frac{1}{(2n+1)^2}\left[\frac{(n+m)!}{(n-m)!}\right]^2 q^n\left[1-e^2\frac{2n+1}{n+1}\frac{n(n+1)-3m^2}{(2n+3)(2n-1)}\right]$$

see *Holota (2001)*.

Moreover, denoting by ψ_{jk} the angular distance of points (β_j, λ_j) and (β_k, λ_k) on a sphere, when β and λ are interpreted as spherical latitude and longitude, respectively, we can apply the well-known ***Legendre addition theorem*** and we arrive at an approximation of $A_{ell}(v_j, v_k)$ given by

$$A^{*}_{ell}(v_j, v_k) = A^{(1)}_{ell}(v_j, v_k) - e^2 A^{(2)}_{ell}(v_j, v_k) \tag{2}$$

where

$$A^{(1)}_{ell}(v_j, v_k) = \frac{4\pi b}{a^2} \sum_{n=0}^{\infty} \frac{n+1}{2n+1} q^n P_n(\cos\psi_{jk})$$

and

$$A^{(2)}_{ell}(v_j, v_k) =$$

$$= \frac{4\pi b}{a^2} \sum_{n=0}^{\infty} \frac{n(n+1)}{(2n+3)(2n-1)} q^n \left[P_n(\cos\psi_{jk}) + \frac{3}{n(n+1)} \frac{\partial^2 P_n(\cos\psi_{jk})}{\partial \lambda_k^2} \right] \tag{3}$$

with

$$\frac{\partial^2 P_n(\cos\psi_{jk})}{\partial\lambda_k^2} = \frac{d^2 P_n(\cos\psi_{jk})}{d(\cos\psi_{jk})^2}\left(\frac{\partial\cos\psi_{jk}}{\partial\lambda_k}\right)^2 + \frac{dP_n(\cos\psi_{jk})}{d\cos\psi_{jk}}\frac{\partial^2\cos\psi_{jk}}{\partial\lambda_k^2} \quad \textbf{(4)}$$

$$\frac{\partial\cos\psi_{jk}}{\partial\lambda_k} = \sin\psi_{jk}\sin\alpha_{jk}\cos\beta_k \quad \textbf{(5)}$$

α_{jk} **denoting the *azimuth* reckoned from the north at the vertex (β_k, λ_k) and**

$$\frac{\partial^2\cos\psi_{jk}}{\partial\lambda_k^2} = \sin\beta_j\sin\beta_k - \cos\psi_{jk} \quad \textbf{(6)}$$

4. Spherical Term

In the following we approach the term $A_{ell}^{(1)}$ and split it as follows

$$A_{ell}^{(1)}(v_j, v_k) = \frac{2\pi b}{a^2} \sum_{n=0}^{\infty} q^n P_n(\cos\psi_{jk}) + \frac{2\pi b}{a^2} S^{(1)},$$

where

$$S^{(1)} = \sum_{n=0}^{\infty} \frac{1}{2n+1} q^n P_n(\cos\psi_{jk})$$

The summation of the first series is easy. It is enough to recall the *definition of Legendre functions* and we immediately see that

$$\sum_{n=0}^{\infty} q^n P_n(\cos\psi_{jk}) = \frac{1}{L} \quad , \quad \text{where} \quad L = \sqrt{1 - 2q\cos\psi_{jk} + q^2}$$

cf. e.g. *Hobson* (*1931*), *Hotine* (*1969*) or *Heiskanen and Moritz* (*1967*).

Thus we have to concentrate on the series in $S^{(1)}$. Its computation leads to an *elliptic integral*.

Indeed, referring to ***Holota and Nesvadba*** (***2014***), we can write that

$$S^{(1)}=\sum_{n=0}^{\infty}\frac{1}{2n+1}q^n P_n(\cos\psi_{jk})=\frac{1}{2}\left(\tan\frac{\varphi}{2}\right)^{-1}\mathcal{F}(k,\varphi) \,,$$

with

$$\mathcal{F}(k,\varphi)=\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

where q was replaced by a new variable $\varphi\in\langle 0,\pi/2\rangle$ according to

$$q=\tan^2(\varphi/2) \quad \text{while} \quad k^2=\cos^2(\psi_{jk}/2)$$

and $\mathcal{F}(k,\varphi)$ is the ***Legendre*** (***incomplete***) ***elliptic integral of the 1st kind***. Thus we have

$$A^{(1)}_{ell}(v_j,v_k)=\frac{2\pi b}{a^2}\frac{1}{L}+\frac{\pi b}{a^2}\left(\tan\frac{\varphi}{2}\right)^{-1}\mathcal{F}(k,\varphi)$$

5. Ellipsoidal Term

For $A_{ell}^{(2)}$ we first apply *Legendre's differential equation*

$$\frac{d^2 P_n(\cos\psi_{jk})}{d(\cos\psi_{jk})^2} = \frac{2\cos\psi_{jk}}{\sin^2\psi_{jk}} \frac{dP_n(\cos\psi_{jk})}{d\cos\psi_{jk}} - \frac{n(n+1)}{\sin^2\psi_{jk}} P_n(\cos\psi_{jk})$$

This in combination with Eqs (3) - (6) yields

$$A_{ell}^{(2)}(v_j, v_k) = \frac{4\pi b}{a^2}\left(a_{jk} S_1^{(2)} + 3b_{jk} S_2^{(2)}\right)$$ **, where**

$$S_1^{(2)} = \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n+3)(2n-1)} q^n P_n(\cos\psi_{jk})$$

$$S_2^{(2)} = \sum_{n=0}^{\infty} \frac{1}{(2n+3)(2n-1)} q^n \frac{dP_n(\cos\psi_{jk})}{d\cos\psi_{jk}}$$

while

$$a_{jk} = 1 - 3\sin^2\alpha_{jk}\cos^2\beta_k$$ **and**

$$b_{jk} = \sin\beta_j \sin\beta_k - (1 - 2\sin^2\alpha_{jk}\cos^2\beta_k)\cos\psi_{jk}$$

5.1. Summation of the $S_1^{(2)}$ Term

For this summation we use the following decomposition

$$S_1^{(2)} = \frac{1}{4L} + \frac{3}{16}\left(A - \frac{1}{q}B\right) \text{ , where } \frac{1}{L} = \sum_{n=0}^{\infty} q^n P_n(\cos\psi_{ik})$$

$$A = \sum_{n=0}^{\infty} \frac{1}{2n-1} q^n P_n(\cos\psi_{ik}) \quad \text{and} \quad B = \sum_{n=0}^{\infty} \frac{1}{2n+3} q^{n+1} P_n(\cos\psi_{ik})$$

Concerning A **, it is easy to verify that**

$$\frac{dA}{dq} - \frac{1}{2q}A = \frac{1}{2qL}$$

which is an elementary differential equation. *Its general solution* is

$$A = C(\psi_{jk})\sqrt{q} + \frac{\sqrt{q}}{2}\int \frac{1}{q\sqrt{qL}}\,dq \text{ ,}$$

where $C(\psi_{jk})$ is an arbitrary function of ψ_{jk}.
In the integral we apply the substitution $q = t^{-1}$ and get

$$\int \frac{1}{q\sqrt{qL}}\,dq = -\int \frac{\sqrt{t}}{\tilde{L}}\,dt \quad \text{with} \quad \tilde{L} = \sqrt{1-2t\cos\psi_{jk} + t^2},$$

which is an *elliptic integral*. To express it in a *trigonometric form* we replace t by a new variable $\tilde{\varphi} \in \langle \pi/2, \pi \rangle$ according to

$$t = \tan^2(\tilde{\varphi}/2).$$

After some algebra we arrive at

$$\int \frac{\sqrt{t}}{\tilde{L}}\,dt = 2\tan\frac{\tilde{\varphi}}{2}\sqrt{1-k^2\sin^2\tilde{\varphi}} + \int \frac{d\tilde{\varphi}}{\sqrt{1-k^2\sin^2\tilde{\varphi}}} - 2\int \sqrt{1-k^2\sin^2\tilde{\varphi}}\,d\tilde{\varphi}$$

with $k^2 = \cos^2(\psi_{jk}/2)$ as above.

Putting now $\varphi = \pi - \tilde{\varphi}$, we can see that $\varphi \in \langle 0, \pi/2 \rangle$ and that $q = \tan^2(\varphi/2)$. Subsequently, one can verify that $C(\psi_{jk}) = 0$ and that

$$A = -\sqrt{1-k^2\sin^2\varphi} + \frac{1}{2}[\mathcal{F}(k,\varphi) - 2\mathcal{E}(k,\varphi)]\tan\frac{\varphi}{2}$$ **with**

$$\mathcal{E}(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

Recall that $\mathcal{F}(k,\varphi)$ is the *Legendre* (*incomplete*) *elliptic integral of the 1st kind* as already introduced above and $\mathcal{E}(k,\varphi)$ is the *Legendre* (*incomplete*) *elliptic integral of the 2nd kind*.

***Concerning* B, we can refer to *Holota and Nesvadba* (*2014*) again and thus can immediately write that**

$$B = \sqrt{1-k^2\sin^2\varphi} + \frac{1}{2}\left(\tan\frac{\varphi}{2}\right)^{-1}[\mathcal{F}(k,\varphi) - 2\mathcal{E}(k,\varphi)]$$

5.2. Summation of the $S_2^{(2)}$ Term

First we recall the following well-known formula

$$\frac{dP_n(\cos\psi_{jk})}{d\cos\psi_{jk}} = (n+1)\frac{\cos\psi_{jk}}{\sin^2\psi_{jk}}P_n(\cos\psi_{jk}) - (n+1)\frac{1}{\sin^2\psi_{jk}}P_{n+1}(\cos\psi_{jk})$$

Hence after some algebra we can deduce that

$$S_2^{(2)} = \frac{1}{\sin^2\psi_{jk}}\left[s'\cos\psi_{jk} - s''\right] \quad \textbf{where}$$

$$s' = \sum_{n=0}^{\infty}\frac{n+1}{(2n+3)(2n-1)}q^n P_n(\cos\psi_{jk}) \quad \textbf{and}$$

$$s'' = \sum_{n=0}^{\infty}\frac{n+1}{(2n+3)(2n-1)}q^n P_{n+1}(\cos\psi_{jk})$$

Moreover, one can verify that the following decomposition holds

$$s' = \frac{1}{8}\left(3A + \frac{1}{q}B\right) \quad \textbf{and similarly} \quad s'' = \frac{1}{8}\left(3s_1'' + s_2''\right),$$

where

$$s_1'' = \sum_{n=0}^{\infty} \frac{1}{2n-1} q^n P_{n+1}(\cos\psi_{jk}) \quad \textbf{and} \quad s_2'' = \sum_{n=0}^{\infty} \frac{1}{2n+3} q^n P_{n+1}(\cos\psi_{jk})$$

Thus it remains to treat the series in the s'' term.

For s_1'', we in addition apply the well know recursion formula

$$P_{n+1}(\cos\psi) = \frac{2n+1}{n+1}\cos\psi \; P_n(\cos\psi) - \frac{n}{n+1} P_{n-1}(\cos\psi)$$

This enables to deduce that

$$s_1'' = \frac{4}{3}\cos\psi_{jk} \sum_{n=0}^{\infty} \frac{1}{2n-1} q^n P_n(\cos\psi_{jk}) + \frac{1}{3}\cos\psi_{ik} \sum_{n=0}^{\infty} \frac{1}{n+1} q^n P_n(\cos\psi_{jk}) -$$

$$-\frac{1}{3}\sum_{n=0}^{\infty} \frac{1}{2n+1} q^{n+1} P_n(\cos\psi_{jk}) - \frac{1}{3}\sum_{n=0}^{\infty} \frac{1}{n+2} q^{n+1} P_n(\cos\psi_{jk})$$

Hence, referring to our results above, to *Holota and Nesvadba* (*2014*) and to *Pick, Pícha and Vyskočil* (*1973*), we have

$$s_1'' = \frac{4}{3} A \cos\psi_{jk} + \frac{1}{3q} \cos\psi_{jk} \ln \frac{L + q - \cos\psi_{jk}}{1 - \cos\psi_{jk}} - \frac{1}{3} q S^{(1)} -$$

$$- \frac{1}{3q} \left(-1 + L + \cos\psi_{jk} \ln \frac{L + q - \cos\psi_{jk}}{1 - \cos\psi_{jk}} \right)$$

or

$$s_1'' = \frac{4}{3} A \cos\psi_{jk} - \frac{q}{3} S^{(1)} - \frac{L - 1}{3q}$$

***For s_2'' the approach is more simple*. We modify the index n and get**

$$s_2'' = \sum_{n=0}^{\infty} \frac{1}{2n+3} q^n P_{n+1}(\cos\psi_{jk}) = -\frac{1}{q} + \frac{1}{q} \sum_{n=0}^{\infty} \frac{1}{2n+1} q^n P_n(\cos\psi_{jk}) = -\frac{1}{q} + \frac{1}{q} S^{(1)}$$

in view of the result in Section 4. *In consequence*

$$s'' = \frac{1}{8} \left(4A \cos\psi_{jk} + \frac{1 - q^2}{q} S^{(1)} - \frac{L}{q} \right)$$

6. Approximation Computation of Galerkin's Element

In this section $A^*_{ell}(v_j, v_k)$ computed on the basis of Eq. (2) and formulas derived in Sections 3 - 5 is *compared* with $A_{ell}(v_j, v_k)$ (exact) computed from Eq. (1). For the ellipsoid of parameters given by GRS80 the difference is illustrated by

$$\delta A_{ell}(v_j, v_k) = \left[A_{ell}(v_j, v_k) - A^*_{ell}(v_j, v_k) \right] \frac{1}{A_{ell}(v_j, v_k)}$$

in *Figs 1 and 2*.

In the figures y_j was taken for the moving point and y_k for the fixed point with $\beta = 0^\circ$ and $\lambda = 0^\circ$. It is assumed that $u_j = u_k = \overline{u} < b$.

One can see the use of $A^*_{ell}(v_j, v_k)$ will enable an *efficient implementation of the weak solution* concept in Earth's gravity field studies.

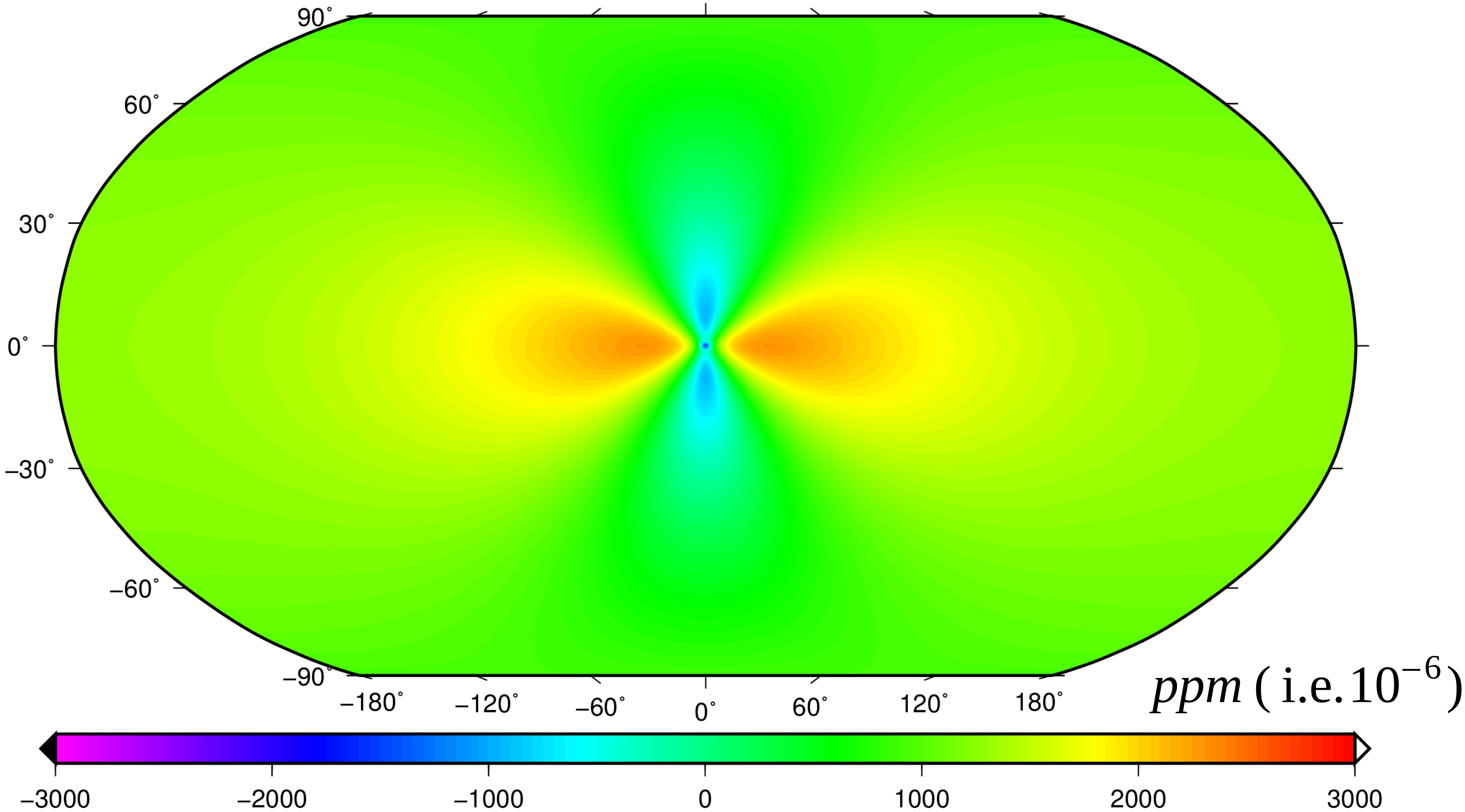

Figure 1. $\delta A_{ell}(v_j, v_k)$ **for** $u_j = u_k = 0.99b$**, i.e.** $\boldsymbol{y}_j$ **and** $\boldsymbol{y}_k$ **ca** $64\,km$ **under the ellipsoid.**

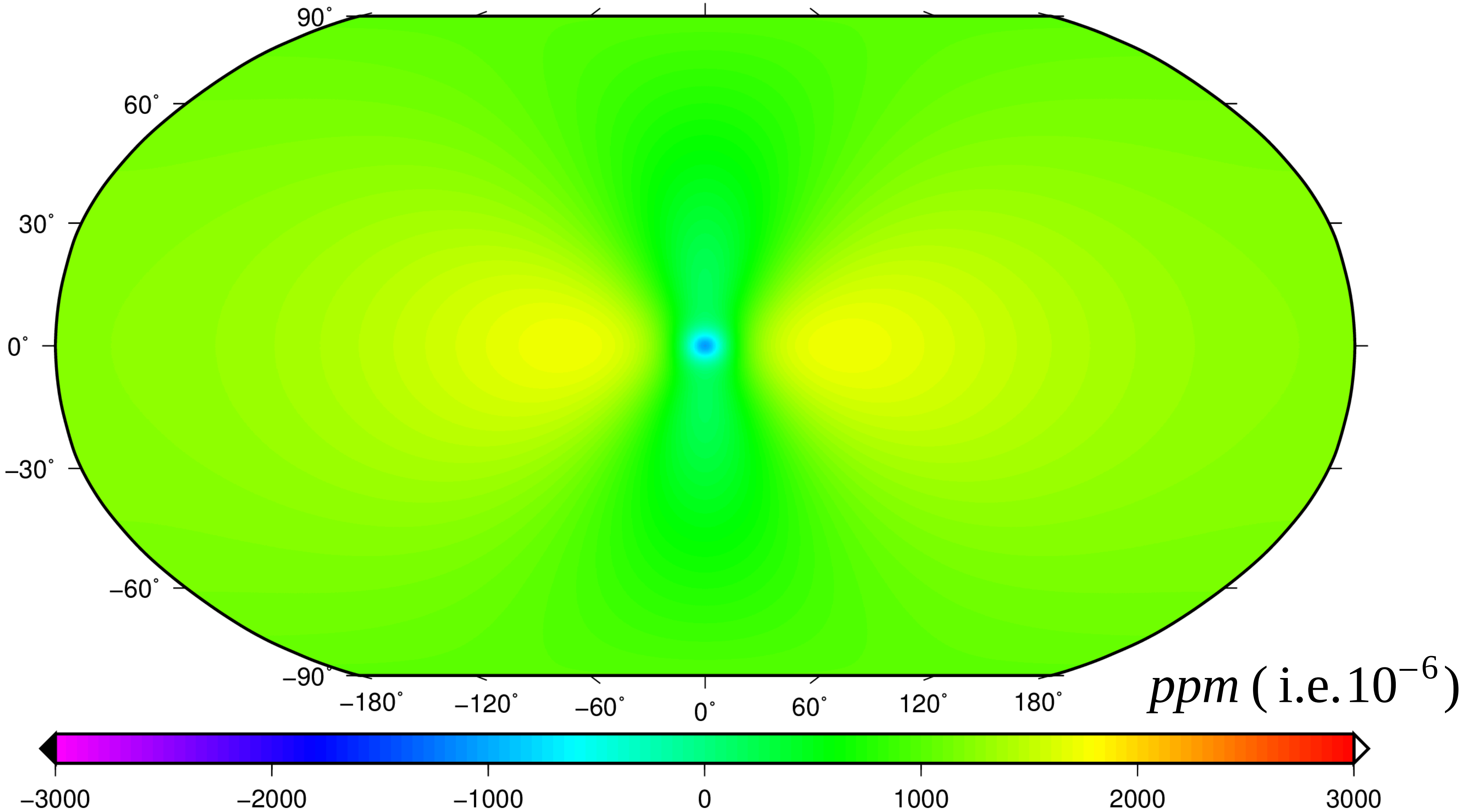

Figure 2. $\delta A_{ell}(v_j, v_k)$ **for** $u_j = u_k = 0.96b$**, i.e.** $\boldsymbol{y}_j$ **and** $\boldsymbol{y}_k$ **ca** $260\,km$ **under the ellipsoid.**

Thank you for your attention !

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